

# A Size Bound for Hamilton Cycles

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## Abstract

Every graph of size  $q$  (the number of edges) and minimum degree  $\delta$  is hamiltonian if  $q \leq \delta^2 + \delta - 1$ . The result is sharp.

## 1 Introduction

The earliest two sufficient conditions for a graph to be hamiltonian are based on three simplest graph invariants, namely order  $n$ , size  $q$  and minimum degree  $\delta$ , in forms of simple algebraic relations between  $n, \delta$  and  $n, q$ , respectively.

**Theorem A** (Dirac, 1952) [3]. Every graph with  $\delta \geq \frac{n}{2}$  is hamiltonian.

**Theorem B** (Erdős and Gallai, 1959) [4]. Every graph with  $q \geq \frac{n^2-3n+5}{2}$  is hamiltonian.

In this paper we present an analogous simple relation between  $\delta$  and  $q$ .

**Theorem 1.** Every graph with  $q \leq \delta^2 + \delta - 1$  is hamiltonian.

The bound  $\delta^2 + \delta - 1$  in Theorem 1 can not be relaxed to  $q \leq \delta^2 + \delta$  since the graph  $K_1 + 2K_\delta$  consisting of two copies of  $K_{\delta+1}$  and having exactly one vertex in common, has  $\delta^2 + \delta$  edges and is not hamiltonian.

## 2 Notations and preliminaries

Only finite undirected graphs without loops or multiple edges are considered. We reserve  $n, q, \delta$  and  $\kappa$  to denote the number of vertices (order), the number of edges (size), the minimum degree and connectivity of a graph. A good reference for any undefined terms is [1].

The set of vertices of a graph  $G$  is denoted by  $V(G)$  and the set of edges by  $E(G)$ . The neighborhood of a vertex  $x \in V(G)$  will be denoted by  $N(x)$ . Set  $d(x) = |N(x)|$ . For a subgraph  $H$  of  $G$  we use  $d_H(x)$  short for  $|N(x) \cap V(H)|$ . Further, we will use  $G_m$  to denote an arbitrary graph on  $m$  vertices.

A simple cycle (or just a cycle)  $C$  of length  $t$  is a sequence  $v_1v_2\dots v_tv_1$  of distinct vertices  $v_1, \dots, v_t$  with  $v_iv_{i+1} \in E(G)$  for each  $i \in \{1, \dots, t\}$ , where  $v_{t+1} = v_1$ . When  $t = 2$ , the cycle  $C = v_1v_2v_1$  on two vertices  $v_1, v_2$  coincides with the edge  $v_1v_2$ , and when  $t = 1$ , the cycle  $C = v_1$  coincides with the vertex  $v_1$ . So, all vertices and edges in a graph can be considered as cycles of lengths 1 and 2, respectively. A graph  $G$  is hamiltonian if  $G$  contains a Hamilton cycle, i.e. a cycle of length  $n$ .

In order to prove the main result we need Theorem A [3] and the following theorem due to Chie [2].

**Theorem C** (Chie, 1980) [2]. Let  $G$  be a 2-connected graph. If  $d(u) + d(v) \geq n - 1$  for each pair of nonadjacent vertices  $u, v$  then either  $G$  is hamiltonian or  $G = G_{(n-1)/2} + \overline{K}_{(n+1)/2}$ .

### 3 Proof of Theorem 1

Assume the converse, that is  $G$  satisfies the condition

$$q \leq \delta^2 + \delta - 1 \tag{1}$$

and is not hamiltonian. Since

$$q = \frac{1}{2} \sum_{u \in V(G)} d(u) \geq \frac{\delta n}{2},$$

we have  $\delta n/2 \leq \delta^2 + \delta - 1$  which is equivalent to

$$\delta \geq \frac{n-1}{2} - \frac{1}{2} + \frac{1}{\delta}.$$

If  $n$  is even, i.e.  $n = 2t$  for some integer  $t$ , then

$$\delta \geq \frac{2t-1}{2} - \frac{1}{2} + \frac{1}{\delta} = t - 1 + \frac{1}{\delta},$$

implying that  $\delta \geq t = n/2$ . By Theorem A,  $G$  is hamiltonian, a contradiction. Let  $n$  is odd, i.e.  $n = 2t + 1$  for some integer  $t$ . Then  $\delta \geq t - 1/2 + 1/\delta$  implying that  $\delta \geq t \geq (n-1)/2$ . Since  $G$  is hamiltonian when  $\delta > (n-1)/2$  (by Theorem A), we can assume that  $\delta = (n-1)/2$ .

Case 1.  $\kappa \geq 2$ .

By Theorem C,  $G = G_{(n-1)/2} + \overline{K}_{(n+1)/2} = G_\delta + \overline{K}_{\delta+1}$ . Clearly  $|E(G)| \geq \delta(\delta + 1)$ , contradicting (1).

Case 2.  $\kappa \leq 1$ .

It follows that  $G$  has a cut vertex  $v$ . Let  $H_1$  and  $H_2$  be any two connected components of  $G \setminus v$ . Denote by  $H_i^*$  the subgraph induced by  $V(H_i) \cup \{v\}$

( $i = 1, 2$ ). Clearly  $|V(H_i^*)| \geq \delta + 1$  ( $i = 1, 2$ ) and  $n \geq 2\delta + 1$ .

Case 2.1.  $xv \notin E(G)$  for some  $x \in V(H_i)$  and  $i \in \{1, 2\}$ .

It follows that  $|V(H_i^*)| \geq \delta + 2$  ( $i = 1, 2$ ) and  $n \geq 2\delta + 2$ . Hence

$$q = \frac{1}{2} \sum_{u \in V(G)} d(u) \geq \frac{\delta n}{2} \geq \delta(\delta + 1),$$

contradicting (1).

Case 2.2.  $xv \in E(G)$  for each  $x \in V(H_i)$  and  $i \in \{1, 2\}$ .

It follows that  $d_{H_i}(v) \geq |V(H_i)| \geq \delta$  for  $i = 1, 2$  implying that  $d(v) \geq d_{H_1}(v) + d_{H_2}(v) \geq 2\delta$ . Hence

$$\begin{aligned} q &= \frac{1}{2} \sum_{u \in V(G)} d(u) = \frac{1}{2} \left( \sum_{u \in V(G) \setminus v} d(u) + d(v) \right) \\ &\geq \frac{1}{2} \sum_{u \in V(G) \setminus v} d(u) + \delta \geq \frac{1}{2}(n-1)\delta + \delta \geq \delta^2 + \delta, \end{aligned}$$

contradicting (1). Theorem 1 is proved.

## References

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